

DIFFEOMORPHISMS OF 7-MANIFOLDS WITH CLOSED G_2 -STRUCTURE

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ABSTRACT. We introduce G_2 vector fields and Rochesterian vector fields on manifolds with a closed G_2 -structure as analogues of symplectic vector fields and Hamiltonian vector fields respectively, and show that the spaces \mathcal{X}_{G_2} of G_2 vector fields and \mathcal{X}_{Roc} of Rochesterian vector fields admit the structure of Lie algebras where the bracket operation is induced from the standard Lie bracket on vector fields; moreover, we define a bracket operation on the space of Rochesterian 1-forms Ω_{Roc}^1 associated to the space of Rochesterian vector fields, giving Ω_{Roc}^1 the structure of a Lie algebra as well. Finally, we show that there is a Lie algebra homomorphism between Rochesterian 1-forms and Rochesterian vector fields and prove a result about its kernel.

INTRODUCTION

The possible holonomy groups for a given 7-dimensional Riemannian manifold includes the exceptional Lie group G_2 by Berger's classification of Riemannian holonomy groups. Such manifolds are called G_2 -manifolds and are equipped with a nondegenerate differential 3-form φ which is torsion-free, $\nabla\varphi = 0$, with respect to the Levi-Civita connection of the metric g_φ defined by φ . This torsion-free condition is equivalent to φ being closed and coclosed, see [17], [19]. Much work has been done to study manifolds with G_2 -holonomy, e. g. [4], [7] and [17], but the condition φ be coclosed is a nonlinear condition since $d^*\varphi = 0$ where d^* depends on the Hodge star defined by the metric g_φ above. If we drop the coclosed condition, then we have a manifold with a *closed G_2 -structure*. Manifolds with closed G_2 -structures have been studied in the articles [5], [6] and [10]; these papers focused predominantly on the metric defined by the nondegenerate closed 3-form φ . We shift our focus to the form φ itself and to results which depend on φ being nondegenerate and closed; this article is a continuation of a project which began with [2] to better understand G_2 -geometry by using the well-established areas of symplectic and contact geometry.

Treating symplectic geometry and G_2 geometry as analogues is not new. [3] and [15] study vector cross products on linear spaces and on manifolds; in particular, it is shown that symplectic geometry is the geometry of 1-fold vector cross products, i. e., almost complex structures, and G_2 geometry is the geometry of 2-fold vector cross products in dimension 7. Further, one can construct, using the metric, a nondegenerate differential form of degree $k + 1$ associated to a k -fold vector cross product. This yields in particular the symplectic form associated to almost complex structures and the G_2 3-form φ associated to 2-fold vector cross products in dimension 7. Examples of manifolds with G_2 structures satisfying various conditions (including closed G_2 structures) are studied in [8], [11], [12] and [14], and classified in [13]. Links between Calabi-Yau geometry and G_2 geometry have been explored in the context of mirror symmetry by Akbulut and Salur [1].

This paper consists of three sections: the first section is a review of ideas from symplectic geometry. We discuss symplectic and Hamiltonian vector fields and show that symplectic vector fields, Hamiltonian vector fields and smooth real-valued functions on M all admit the structure of Lie algebras with Lie bracket on the symplectic and Hamiltonian vector fields induced from the Lie bracket structure on the space of all vector fields and the Lie bracket on smooth real-valued functions on M given by a Poisson bracket; further, there is a Lie algebra homomorphism between the Lie algebra of smooth real-valued functions on M and the Lie algebra of the Hamiltonian vector fields. We give a brief introduction to G_2 geometry in the second section; in the third section, we define analogues of symplectic and Hamiltonian vector fields given by G_2 and Rochesterian vector fields respectively, and prove the following results for Rochesterian vector fields:

Theorem. *Let (M_1, φ_1) and (M_2, φ_2) be two manifolds with closed G_2 -structures φ_1, φ_2 respectively. Let $\pi_i : M_1 \times M_2 \rightarrow M_i$ be the standard projection map, and define $\tilde{\varphi} := \pi_1^* \varphi_1 - \pi_2^* \varphi_2$, a closed 3-form on the product manifold $M_1 \times M_2$. A diffeomorphism $\Upsilon : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ is a G_2 -morphism if and only if $\tilde{\varphi}|_{\Gamma_\Upsilon} \equiv 0$, where $\Gamma_\Upsilon := \{(p, \Upsilon(p)) \in M_1 \times M_2 : p \in M_1\}$ is the graph of Υ in $M_1 \times M_2$.*

Theorem. *There are no Rochesterian vector fields on a closed manifold M with closed G_2 -structure φ .*

Theorem. *Every Rochesterian vector field on a manifold M with closed G_2 -structure φ is a G_2 vector field. If $H^2(M) = \{0\}$, then every G_2 vector field on a manifold with closed G_2 -structure is a Rochesterian vector field.*

We next show that the spaces of G_2 and Rochesterian vector fields as well as the associated space of Rochesterian 1-forms admit the structure of Lie algebras with Lie bracket on the G_2 and Rochesterian vector fields induced from the Lie bracket structure on the space of all vector fields. In particular, we show

Theorem. *For any G_2 vector fields X_1, X_2 , $[X_1, X_2]$ is a Rochesterian vector field with associated Rochesterian 1-form given by $\varphi(X_2, X_1, \cdot)$.*

Finally, we show there exists a Lie algebra homomorphism from the Lie algebra of Rochesterian 1-forms to the Lie algebra of Rochesterian vector fields, and prove a result relating the Lie algebra of Rochesterian 1-forms on \mathbb{R}^7 to G_2 -morphisms of \mathbb{R}^7 .

Theorem. (1) *Let $\Phi : (\Omega_{Roc}^1(M), \{\cdot, \cdot\}) \rightarrow (\mathcal{X}_{Roc}, [\cdot, \cdot])$ be the Lie algebra homomorphism. Given two Rochesterian 1-forms $\alpha_1, \alpha_2 \in \Omega_{Roc}^1(M)$, $\{\alpha_1, \alpha_2\} \in \ker \Phi$ if and only if $d\alpha_1$ is constant along the flow lines of X_{α_2} .*
 (2) *If $\psi : \mathbb{R}^7 \rightarrow \mathbb{R}^7$ is a G_2 -morphism then $\psi^*(\{\alpha, \tau\}) = \{\psi^*\alpha, \psi^*\tau\}$.*

1. HAMILTONIAN VECTOR FIELDS AND SYMPLECTIC VECTOR FIELDS

The material in this section comes from [18] and [20]. Let (M, ω) be an arbitrary $2n$ -dimensional manifold with a closed, nondegenerate 2-form ω .

Definition 1.1. (1) A vector field Y is called a *symplectic vector field* if the flow induced by Y preserves the symplectic form ω , that is, Y is a symplectic vector field if and only if $\mathcal{L}_Y \omega = 0$.
 (2) A vector field Y is called a *Hamiltonian vector field* if there exists a smooth real-valued function H on M such that $Y \lrcorner \omega = dH$.

Notice that since ω is closed, we have, by the Cartan Formula, $\mathcal{L}_Y\omega = d(Y \lrcorner \omega) + Y \lrcorner d\omega = d(Y \lrcorner \omega)$ for any vector field Y , so Y is symplectic if and only if the 1-form $Y \lrcorner \omega$ is closed. Hamiltonian vector fields always exist since ω induces an isomorphism of the tangent bundle and the cotangent bundle: Given any smooth function H , dH is a covector field, so there exists a unique vector field Y_H such that $Y_H \lrcorner \omega = dH$. An immediate consequence of these definitions is that a Hamiltonian vector field Y is always symplectic for if $Y \lrcorner \omega = dH$ for some smooth real-valued function H , then $d(Y \lrcorner \omega) = d(dH) = 0$. The converse is not true in general; in fact, the obstruction for a symplectic vector field Y on (M, ω) to be Hamiltonian is $H^1(M)$. Indeed, if $H^1(M) = \{0\}$, i. e., every closed 1-form is exact, then for a symplectic vector field Y , there exists a smooth real-valued function H on M such that $Y \lrcorner \omega = dH$, that is, Y is a Hamiltonian vector field.

Let $\mathcal{X}(M)$ denote the space of vector fields on M , $\mathcal{X}_{\text{symp}}(M)$ the subspace of symplectic vector fields on M and $\mathcal{X}_{\text{Ham}}(M)$ the subspace of Hamiltonian vector fields on M , and equip $\mathcal{X}(M)$ with the standard Lie bracket $[X, Y] = XY - YX$:

Proposition 1.2. *If Y_1, Y_2 are symplectic vector fields, then the Lie bracket $[Y_1, Y_2]$ is a Hamiltonian vector field.*

Proof. For an arbitrary differential form τ we have $[Y, \tilde{Y}] \lrcorner \tau = \mathcal{L}_Y(\tilde{Y} \lrcorner \tau) - \tilde{Y} \lrcorner (\mathcal{L}_Y \tau)$, so

$$\begin{aligned} [Y_1, Y_2] \lrcorner \omega &= \mathcal{L}_{Y_1}(Y_2 \lrcorner \omega) + Y_2 \lrcorner (\underbrace{\mathcal{L}_{Y_1} \omega}_{=0}) = \mathcal{L}_{Y_1}(Y_2 \lrcorner \omega) \\ &= d(Y_1 \lrcorner Y_2 \lrcorner \omega) + Y_1 \lrcorner (\underbrace{d(Y_2 \lrcorner \omega)}_{=0}) = d(\omega(Y_2, Y_1)). \end{aligned}$$

Hence $[Y_1, Y_2]$ is a Hamiltonian vector field with generating Hamiltonian function $\omega(Y_2, Y_1)$. \square

Corollary 1.3. *The subspaces $\mathcal{X}_{\text{symp}}(M)$ and $\mathcal{X}_{\text{Ham}}(M)$ of $\mathcal{X}(M)$ are closed under the Lie bracket operation inherited from $\mathcal{X}(M)$; hence, there are the following inclusions of Lie algebras:*

$$(\mathcal{X}_{\text{Ham}}(M), [\cdot, \cdot]) \subseteq (\mathcal{X}_{\text{symp}}(M), [\cdot, \cdot]) \subseteq (\mathcal{X}(M), [\cdot, \cdot]).$$

We now focus on the real-valued smooth functions on M , $C^\infty(M)$. For $f \in C^\infty(M)$, the assignment $f \mapsto X_f$ where X_f is the associated Hamiltonian vector field is linear. Given $f, g \in C^\infty(M)$, then

$$(X_f + X_g) \lrcorner \omega = (X_f \lrcorner \omega) + (X_g \lrcorner \omega) = df + dg = d(f + g) = X_{f+g} \lrcorner \omega,$$

so that by nondegeneracy of ω , we have $X_{f+g} = X_f + X_g$. Similarly, $X_{af} = aX_f$. We now equip $C^\infty(M)$ with a bracket operation as follows: For $f, g \in C^\infty(M)$, define $\{f, g\} = \omega(X_f, X_g) \in C^\infty(M)$. Consider the Hamiltonian vector field $X_{\{f, g\}}$:

$$X_{\{f, g\}} \lrcorner \omega = X_{\omega(X_f, X_g)} \lrcorner \omega = ([X_g, X_f]) \lrcorner \omega,$$

so that $X_{\{f, g\}} = -[X_f, X_g]$.

Proposition 1.4. *The bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ satisfies the Jacobi identity.*

Proof.

$$\begin{aligned} X_{\{\{f, g\}, h\}} + X_{\{\{g, h\}, f\}} + X_{\{\{h, f\}, g\}} &= -[X_{\{f, g\}}, X_h] - [X_{\{g, h\}}, X_f] - [X_{\{h, f\}}, X_g] \\ &= -[-[X_f, X_g], X_h] - [-[X_g, X_h], X_f] - [-[X_h, X_f], X_g] \end{aligned}$$

$$= [[X_f, X_g], X_h] + [[X_g, X_h], X_f] + [[X_h, X_f], X_g] = 0.$$

□

Hence, $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra, and there is a Lie algebra homomorphism $\Psi : (C^\infty(M), \{\cdot, \cdot\}) \rightarrow (\mathcal{X}_{Ham}, [\cdot, \cdot])$ given by $f \mapsto X_f$. Assume that $\Psi(f) = X_f = 0$, then $0 = X_f \lrcorner \omega = df$ implies that f is locally constant (or constant if M is connected); therefore, $\ker \Psi = \{(\text{locally}) \text{ constant functions on } M\}$, so every Hamiltonian vector field is defined by a smooth real-valued function on M which is unique up to the addition of a locally constant smooth function.

Theorem 1.5. (1) *For $f, g \in C^\infty(M)$, $\{f, g\} = 0$ if and only if f is constant along the integral curves ψ_t determined by X_g .*
 (2) *If $\psi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a symplectomorphism then $\{f, g\} \circ \psi = \{f \circ \psi, g \circ \psi\}$ for all $f, g \in C^\infty(\mathbb{R}^{2n})$.*

Proof. (1)

$$\begin{aligned} \frac{d}{dt}(f \circ \psi_t) &= \psi_t^* \mathcal{L}_{X_g} f = \psi_t^*(X_g \lrcorner df) \\ &= \psi_t^*(X_g \lrcorner X_f \lrcorner \omega) = \psi_t^* \omega(X_f, X_g) = \psi_t^* \{f, g\}. \end{aligned}$$

(2) Note that for $p \in \mathbb{R}^{2n}$, we have the maps

$$\begin{aligned} d\psi_p : T_p \mathbb{R}^{2n} &\rightarrow T_{\psi(p)} \mathbb{R}^{2n}; \\ \psi_p^* : T_{\psi(p)}^* \mathbb{R}^{2n} &\rightarrow T_p^* \mathbb{R}^{2n}; \\ d\psi_{\psi(p)}^{-1} &= (d\psi_p)^{-1} : T_{\psi(p)} \mathbb{R}^{2n} \rightarrow T_p \mathbb{R}^{2n}. \end{aligned}$$

Since $\psi^* \omega = \omega$ and $(\psi^{-1})^* \omega = \omega$, these equations yield

$$\begin{aligned} \omega_p(\cdot, \cdot) &= \psi_p^*(\omega_{\psi(p)})(\cdot, \cdot) = \omega_{\psi(p)}(d\psi_p \cdot, d\psi_p \cdot); \\ \omega_{\psi(p)}(\cdot, \cdot) &= (\psi_{\psi(p)}^{-1})^*(\omega_p)(\cdot, \cdot) = \omega_p(d\psi_{\psi(p)}^{-1} \cdot, d\psi_{\psi(p)}^{-1} \cdot). \end{aligned}$$

Thus, for a function $f \in C^\infty(\mathbb{R}^{2n})$ and vector field Y on \mathbb{R}^{2n} , we calculate

$$\begin{aligned} (X_{f \circ \psi} \lrcorner \omega)_p(Y_p) &= d(f \circ \psi)_p(Y_p) = df_{\psi(p)}(d\psi_p Y_p) = \psi_p^*(df_{\psi(p)})(Y_p) \\ &= \psi_p^*((X_f \lrcorner \omega)_{\psi(p)})(Y_p) = \psi_p^*(\omega_{\psi(p)}((X_f)_{\psi(p)}, \cdot))(Y_p) = \omega_{\psi(p)}((X_f)_{\psi(p)}, d\psi_p Y_p) \\ &= \omega_p(d\psi_{\psi(p)}^{-1}(X_f)_{\psi(p)}, d\psi_{\psi(p)}^{-1}(d\psi_p Y_p)) = \omega_p((d\psi^{-1})_{\psi(p)}(X_f)_{\psi(p)}, Y_p), \end{aligned}$$

that is, $(X_{f \circ \psi})_p = (d\psi^{-1})_{\psi(p)}(X_f)_{\psi(p)}$. Hence we find that

$$\begin{aligned} (\{f, g\} \circ \psi)(p) &= \omega(X_f, X_g)(\psi(p)) = \omega_{\psi(p)}((X_f)_{\psi(p)}, (X_g)_{\psi(p)}) \\ &= \omega_p(d\psi_{\psi(p)}^{-1}(X_f)_{\psi(p)}, d\psi_{\psi(p)}^{-1}(X_g)_{\psi(p)}) = \omega_p((X_{f \circ \psi})_p, (X_{g \circ \psi})_p) \\ &= \omega(X_{f \circ \psi}, X_{g \circ \psi})(p) = \{f \circ \psi, g \circ \psi\}(p). \end{aligned}$$

□

2. G_2 GEOMETRY

References for this section include [16] and [17].

The octonions \mathbb{O} is an 8-dimensional real normed algebra equipped with the standard Euclidean inner product. Further, there is a cross product operation given by $u \times v = \text{Im}(\bar{v}u)$ where \bar{v} is the conjugate of v for $u, v \in \mathbb{O}$. This is an alternating form on $\text{Im } \mathbb{O}$ since, for any $u \in \text{Im } \mathbb{O}$, $u^2 \in \text{Re } \mathbb{O}$. We now define a 3-form on $\text{Im } \mathbb{O}$ by $\varphi(u, v, w) = \langle u \times v, w \rangle$. In terms of the standard orthonormal basis $\{e_1, \dots, e_7\}$

(with dual basis $\{e^1, \dots, e^7\}$) of $\text{Im } \mathbb{O}$, $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$ where $e^{ijk} = e^i \wedge e^j \wedge e^k$. Under the isomorphism $\mathbb{R}^7 \simeq \text{Im } \mathbb{O}$, with coordinates on \mathbb{R}^7 given by (x^1, \dots, x^7) , we have $\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}$.

Definition 2.1. Let M be a 7-dimensional manifold. M has a G_2 -structure if there is a smooth 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$, the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. M has a *closed* G_2 -structure if the 3-form φ is also closed, $d\varphi = 0$.

Equivalently, a smooth 7-dimensional manifold M has a G_2 -structure if its tangent frame bundle reduces to a G_2 -bundle. For a manifold with G_2 -structure φ , there is a natural Riemannian metric and orientation induced by φ given by $(Y \lrcorner \varphi) \wedge (\tilde{Y} \lrcorner \varphi) \wedge \varphi = \langle Y, \tilde{Y} \rangle_\varphi d\text{vol}_M$. In particular, the 3-form φ is nondegenerate.

Definition 2.2. Let (M_1, φ_1) and (M_2, φ_2) be 7-manifolds with G_2 -structures. If $\Upsilon : M_1 \rightarrow M_2$ is a diffeomorphism such that $\Upsilon^*(\varphi_2) = \varphi_1$, then Υ is called a G_2 -morphism and (M_1, φ_1) , (M_2, φ_2) are said to be G_2 -morphic.

Notice that given a G_2 -morphism $\Upsilon : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$, $d\varphi_1 = 0$ if and only if $d\varphi_2 = 0$ since d commutes with pullback maps.

Let (M_1, φ_1) and (M_2, φ_2) be two 7-dimensional manifolds with closed G_2 -structures. Let $M_1 \times M_2$ be the standard Cartesian product of M_1 and M_2 with canonical projection maps $\pi_i : M_1 \times M_2 \rightarrow M_i$. Define a 3-form $\varphi = \pi_1^* \varphi_1 + \pi_2^* \varphi_2$. This form is closed since

$$d\varphi = d\pi_1^* \varphi_1 + d\pi_2^* \varphi_2 = \pi_1^* d\varphi_1 + \pi_2^* d\varphi_2 = 0.$$

Indeed, for any $a_1, a_2 \in \mathbb{R}$, $a_1 \pi_1^* \varphi_1 + a_2 \pi_2^* \varphi_2$ defines a closed 3-form on $M_1 \times M_2$. Taking $a_1 = 1$ and $a_2 = -1$, we have the closed 3-form $\tilde{\varphi} = \pi_1^* \varphi_1 - \pi_2^* \varphi_2$.

Theorem 2.3. A diffeomorphism $\Upsilon : (M_1, \varphi_1) \rightarrow (M_2, \varphi_2)$ is a G_2 -morphism if and only if $\tilde{\varphi}|_{\Gamma_\Upsilon} \equiv 0$, where $\Gamma_\Upsilon := \{(p, \Upsilon(p)) \in M_1 \times M_2 : p \in M_1\}$.

Proof. The submanifold Γ_Υ is the embedded image of M_1 in $M_1 \times M_2$ with embedding given by $\tilde{\Upsilon} : M_1 \rightarrow M_1 \times M_2$ $\tilde{\Upsilon}(p) = (p, \Upsilon(p))$. Then $\tilde{\varphi}|_{\Gamma_\Upsilon} = 0$ if and only if

$$\begin{aligned} 0 &= \tilde{\Upsilon}^* \tilde{\varphi} = \tilde{\Upsilon}^* \pi_1^* \varphi_1 - \tilde{\Upsilon}^* \pi_2^* \varphi_2 \\ &= (\pi_1 \circ \tilde{\Upsilon})^* \varphi_1 - (\pi_2 \circ \tilde{\Upsilon})^* \varphi_2 = (id_{M_1})^* \varphi_1 - \Upsilon^* \varphi_2 = \varphi_1 - \Upsilon^* \varphi_2. \end{aligned}$$

□

3. G_2 VECTOR FIELDS AND ROCHESTERIAN VECTOR FIELDS

Definition 3.1. Let (M, φ) be a manifold with closed G_2 -structure.

- (1) A vector field X is called a *Rochesterian vector field* if there exists a 1-form α , called a *Rochesterian 1-form*, such that $X \lrcorner \varphi = d\alpha$.
- (2) A vector field X is called a G_2 vector field if the flow induced by X preserves the G_2 -structure; equivalently, X is a G_2 vector field if $\mathcal{L}_X \varphi = 0$.

As in the symplectic case, we have $d(x \lrcorner \varphi) = 0$ since $0 = \mathcal{L}_X \varphi = d(X \lrcorner \varphi) + X \lrcorner d\varphi = d(X \lrcorner \varphi)$; in contrast, the existence of Rochesterian vector fields is a bit more delicate than that of Hamiltonian vector fields as is seen as a consequence of the following theorem (see [2, Theorem 2.4] for the original form of the statement and proof of this theorem):

Theorem 3.2. *Let M be a closed manifold, and let φ be a closed G_2 -structure on M . Then for any vector field X on M , $X \lrcorner \varphi$ is not exact, i. e., there does not exist a 1-form α such that $X \lrcorner \varphi = d\alpha$; therefore, there are no Rochesterian vector fields on a closed manifold M with closed G_2 -structure φ .*

Proof. Assume on the contrary that X is a vector field such that there exists a 1-form α with $X \lrcorner \varphi = d\alpha$. Then using the G_2 -metric defined by φ , we have that

$$\begin{aligned} \langle X, X \rangle d\text{vol}_M &= (X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \varphi = d\alpha \wedge d\alpha \wedge \varphi \\ &= d(\alpha \wedge d\alpha \wedge \varphi). \end{aligned}$$

Assuming the X is normalized to have length 1, we have

$$0 < \text{vol}(M) = \int_M d\text{vol}_M = \int_M d(\alpha \wedge d\alpha \wedge \varphi) = \int_{\partial M} \alpha \wedge d\alpha \wedge \varphi = 0$$

by Stokes' Theorem and the fact that $\partial M = \emptyset$ giving a contradiction. \square

In general, the nondegeneracy of φ is not enough to guarantee that given any 1-form α , there is a vector field X_α such that $X_\alpha \lrcorner \varphi = d\alpha$. In fact, the nondegeneracy only tells us that the map $\tilde{\varphi}$ from vector fields into 2-forms given by $\tilde{\varphi}(X) = X \lrcorner \varphi$ is injective. The above theorem tells us that $d\Omega^1(M) \cap \text{im}(\tilde{\varphi}) = \{0\}$ for a closed manifold M with closed G_2 -structure. Hence, we assume from now on that M is either noncompact or that M is compact with nonempty boundary. In the case of $(\mathbb{R}^7, \varphi_0)$ simple calculations show that every coordinate vector field is a Rochesterian vector field. If X is a 3-dimensional manifold, then in the case of $(T^*X \times \mathbb{R}, \varphi = \text{Re } \Omega + \omega \wedge dt)$, the vector field $\frac{\partial}{\partial t}$ is Rochesterian with associated Rochesterian 1-form given by the tautological 1-form α on T^*X (see [9]).

Theorem 3.3. *Every Rochesterian vector field on a manifold M with closed G_2 -structure φ is a G_2 vector field. If $H^2(M) = \{0\}$, then every G_2 vector field on a manifold with closed G_2 -structure is a Rochesterian vector field.*

Proof. The first statement follows immediately from the definitions. Let X be a vector field such that $X \lrcorner \varphi = d\alpha$ for some 1-form α ; then $\mathcal{L}_X \varphi = d(X \lrcorner \varphi) = d(d\alpha) = 0$. Now, X is a G_2 vector field implies that $X \lrcorner \varphi$ is a closed 2-form; $H^2(M) = \{0\}$ implies that every closed 2-form is exact, so there exists a 1-form α with $X \lrcorner \varphi = d\alpha$. \square

As in the symplectic case, let $\mathcal{X}(M)$ denote the set of vector fields on M ; let $\mathcal{X}_{G_2}(M)$ denote the set of G_2 vector fields on M ; finally, let $\mathcal{X}_{Roc}(M)$ denote the set of Rochesterian vector fields on M . That \mathcal{X}_{G_2} and \mathcal{X}_{Roc} are closed under addition and scalar multiplication follows by the linearity of d and the interior product.

Proposition 3.4. *For any G_2 vector fields X_1, X_2 , there exists a 1-form α such that $[X_1, X_2] \lrcorner \varphi = d\alpha$.*

Proof. The result is a direct consequence of the following calculation:

$$\begin{aligned} [X_1, X_2] \lrcorner \varphi &= \mathcal{L}_{X_1}(X_2 \lrcorner \varphi) + X_2 \lrcorner (\underbrace{\mathcal{L}_{X_1} \varphi}_{=0}) = \mathcal{L}_{X_1}(X_2 \lrcorner \varphi) \\ &= d(X_1 \lrcorner X_2 \lrcorner \varphi) + X_1 \lrcorner (\underbrace{d(X_2 \lrcorner \varphi)}_{=0}) = d(\varphi(X_2, X_1, \cdot)). \end{aligned}$$

Thus, $[X_1, X_2]$ is a Rochesterian vector field with generating 1-form given by $\varphi(X_2, X_1, \cdot)$. \square

Thus, we have the following inclusions of Lie algebras:

$$(\mathcal{X}_{Roc}(M), [\cdot, \cdot]) \subseteq (\mathcal{X}_{G_2}(M), [\cdot, \cdot]) \subseteq (\mathcal{X}(M), [\cdot, \cdot]).$$

Let $\Omega_{Roc}^1(M)$ be the set of Rochesterian 1-forms. Let $\alpha, \tau \in \Omega_{Roc}^1(M)$, then $\alpha + \tau \in \Omega_{Roc}^1(M)$ since $d(\alpha + \tau) = d\alpha + d\tau = X_\alpha \lrcorner \varphi + X_\tau \lrcorner \varphi = (X_\alpha + X_\tau) \lrcorner \varphi$. In this way, we see that $\alpha + \tau$ is a Rochesterian 1-form with Rochesterian vector field given by $X_{\alpha+\tau} := X_\alpha + X_\tau$; similarly, for $a \in \mathbb{R}$, $a\alpha$ is Rochesterian 1-form with Rochesterian vector field given by $X_{a\alpha} := aX_\alpha$, so $\Omega_{Roc}^1(M)$ is a vector space. From this, we see that, for a Rochesterian 1-form α , the assignment $\alpha \mapsto X_\alpha$ where X_α is the associated Rochesterian vector field is linear. We now equip $\Omega_{Roc}^1(M)$ with a bracket as follows: for $\alpha, \tau \in \Omega_{Roc}^1(M)$, define $\{\alpha, \tau\} = \varphi(X_\alpha, X_\tau, \cdot)$. $\{\alpha, \tau\} \in \Omega_{Roc}^1(M)$ with Rochesterian vector field given by $[X_\tau, X_\alpha]$ because $d(\{\alpha, \tau\}) = d(\varphi(X_\alpha, X_\tau, \cdot)) = [X_\tau, X_\alpha] \lrcorner \varphi$. Using this fact, the bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity:

$$\begin{aligned} X_{\{\alpha, \tau\}, v} + X_{\{\tau, v\}, \alpha} + X_{\{v, \alpha\}, \tau} &= -[X_{\{\alpha, \tau\}}, X_v] - [X_{\{\tau, v\}}, X_\alpha] - [X_{\{v, \alpha\}}, X_\tau] \\ &= -[-[X_\alpha, X_\tau], X_v] - [-[X_\tau, X_v], X_\alpha] - [-[X_v, X_\alpha], X_\tau] \\ &= [[X_\alpha, X_\tau], X_v] + [[X_\tau, X_v], X_\alpha] + [[X_v, X_\alpha], X_\tau] = 0. \end{aligned}$$

Hence, $(\Omega_{Roc}^1(M), \{\cdot, \cdot\})$ is a Lie algebra, and there is a Lie algebra homomorphism $\Phi : (\Omega_{Roc}^1(M), \{\cdot, \cdot\}) \rightarrow (\mathcal{X}_{Roc}, [\cdot, \cdot])$. Finally, assume that $\Phi(\alpha) = X_\alpha = 0$, then $0 = X_\alpha \lrcorner \varphi = d\alpha$ which implies that α is a closed 1-form. Hence, Rochesterian vector fields are uniquely defined by their Rochesterian 1-forms, up to the addition of a closed 1-form.

Theorem 3.5. (1) *Given two Rochesterian 1-forms $\alpha_1, \alpha_2 \in \Omega_{Roc}^1(M)$, $\{\alpha_1, \alpha_2\} \in \ker \Phi$ if and only if $d\alpha_1$ is constant along the flow lines of X_{α_2} .*
 (2) *If $\psi : \mathbb{R}^7 \rightarrow \mathbb{R}^7$ is a G_2 -morphism then $\psi^*(\{\alpha, \tau\}) = \{\psi^*\alpha, \psi^*\tau\}$ for all $\alpha, \tau \in \Omega_{Roc}^1(\mathbb{R}^7)$.*

Proof. (1) From the definition of the bracket, we have

$$\begin{aligned} \{\alpha_1, \alpha_2\} &= \varphi(X_{\alpha_1}, X_{\alpha_2}, \cdot) = X_{\alpha_2} \lrcorner (X_{\alpha_1} \lrcorner \varphi) \\ &= X_{\alpha_2} \lrcorner d\alpha_1 = \mathcal{L}_{X_{\alpha_2}} \alpha_1 - d(X_{\alpha_2} \lrcorner \alpha_1). \end{aligned}$$

From this, we see that $d\{\alpha_1, \alpha_2\} = d\mathcal{L}_{X_{\alpha_2}} \alpha_1 = \mathcal{L}_{X_{\alpha_2}}(d\alpha_1)$. Then $\{\alpha_1, \alpha_2\} \in \ker \Phi$ if and only if $X_{\{\alpha_1, \alpha_2\}} = 0$ if and only if $\mathcal{L}_{X_{\alpha_2}}(d\alpha_1) = d\{\alpha_1, \alpha_2\} = X_{\{\alpha_1, \alpha_2\}} \lrcorner \varphi = 0$.

(2) Note that for $p \in \mathbb{R}^7$, we have the maps

$$\begin{aligned} d\psi_p &: T_p \mathbb{R}^7 \rightarrow T_{\psi(p)} \mathbb{R}^7; \\ \psi_p^* &: T_{\psi(p)}^* \mathbb{R}^7 \rightarrow T_p^* \mathbb{R}^7; \\ d\psi_p^{-1} &= (d\psi_p)^{-1} : T_{\psi(p)} \mathbb{R}^7 \rightarrow T_p \mathbb{R}^7. \end{aligned}$$

If ψ is a G_2 -morphism, then $\psi^*\varphi = \varphi$ and $(\psi^{-1})^*\varphi = \varphi$. By definition, these equations yield, for $p \in \mathbb{R}^7$,

$$\varphi_p(\cdot, \cdot, \cdot) = \psi_p^*(\varphi_{\psi(p)})(\cdot, \cdot, \cdot) = \varphi_{\psi(p)}(d\psi_p \cdot, d\psi_p \cdot, d\psi_p \cdot)$$

and

$$\varphi_{\psi(p)}(\cdot, \cdot, \cdot) = (\psi_{\psi(p)}^{-1})^*(\varphi_p)(\cdot, \cdot, \cdot) = \varphi_p(d\psi_{\psi(p)}^{-1} \cdot, d\psi_{\psi(p)}^{-1} \cdot, d\psi_{\psi(p)}^{-1} \cdot).$$

Thus, we calculate for a Rochesterian 1-form $\alpha \in \Omega_{Roc}^1(M)$ and vector fields Y, Z on \mathbb{R}^7 ,

$$\begin{aligned} (X_{\psi^*\alpha} \lrcorner \varphi)_p(Y_p, Z_p) &= d(\psi^*\alpha)_p(Y_p, Z_p) = \psi_p^*(d\alpha_{\psi(p)})(Y_p, Z_p) \\ &= \psi_p^*((X_\alpha \lrcorner \varphi)_{\psi(p)})(Y_p, Z_p) = \psi_p^*(\varphi_{\psi(p)}((X_\alpha)_{\psi(p)}, \cdot, \cdot))(Y_p, Z_p) \\ &= \varphi_{\psi(p)}((X_\alpha)_{\psi(p)}, d\psi_p Y_p, d\psi_p Z_p) = \varphi_{\psi(p)}((X_\alpha)_{\psi(p)}, \tilde{Y}_{\psi(p)}, \tilde{Z}_{\psi(p)}) \\ &= \varphi_p((d\psi^{-1})_{\psi(p)}(X_\alpha)_{\psi(p)}, (d\psi^{-1})_{\psi(p)}(d\psi_p Y_p), (d\psi^{-1})_{\psi(p)}(d\psi_p Z_p)) \\ &= \varphi_p((d\psi^{-1})_{\psi(p)}(X_\alpha)_{\psi(p)}, Y_p, Z_p), \end{aligned}$$

that is, $(X_{\psi^*\alpha})_p = (d\psi^{-1})_{\psi(p)}(X_\alpha)_{\psi(p)}$. Hence we find that

$$\begin{aligned} (\psi^*\{\alpha, \tau\})_p(Y_p) &= (\psi^*\varphi(X_\alpha, X_\tau))_p(Y_p) \\ &= \psi_p^*(\varphi_{\psi(p)}((X_\alpha)_{\psi(p)}, (X_\tau)_{\psi(p)}, \cdot))(Y_p) = \varphi_{\psi(p)}((X_\alpha)_{\psi(p)}, (X_\tau)_{\psi(p)}, d\psi_p Y_p) \\ &= \varphi_p(d\psi_{\psi(p)}^{-1}(X_\alpha)_{\psi(p)}, d\psi_{\psi(p)}^{-1}(X_\tau)_{\psi(p)}, d\psi_{\psi(p)}^{-1}(d\psi_p Y_p)) = \varphi_p((X_{\psi^*\alpha})_p, (X_{\psi^*\tau})_p, Y_p) \\ &= \varphi_p((X_{\psi^*\alpha})_p, (X_{\psi^*\tau})_p, \cdot)(Y_p) = \{\psi^*\alpha, \psi^*\tau\}_p(Y_p). \end{aligned}$$

□

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